True or false?

1. If a system of linear equations has more equations than unknown variables then there is no solution.

2. If a system has fewer equations than variables there is always a solution.

3. If a system has fewer equations than variables and there is at least one solution then there are infinitely many solutions.

Pre-class Warm-up!!!

Do you already know how to do the following matrix multiplications?

a. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Yes / No

b. $\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Yes / No

c. $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$ Yes / No

d. $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ Yes / No

Section 3.4. Matrix operations

We learn (for matrices) what the following mean:

A = B, A+B, A-B, cA, AB
Rules of matrix algebra
Identity matrix, inverse matrix
(In an exercise at the end) the trace of a square matrix, the determinant of a 2 x 2 matrix.

Two matrices A and B are equal $\Longleftrightarrow$ in each position their entries are equal. This implies: if A is $m \times n$ then B is $m \times n$ also.

If A, B have the same size we can add them, thus

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$$

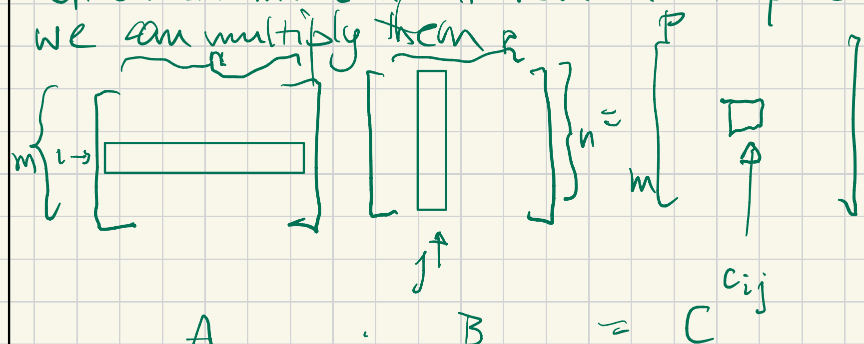
$$2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

Start by multiplying (row) $\cdot$ (column)

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1) + 2\cdot 2 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} au + bv + cw \end{bmatrix}$$

Given an $m \times n$ matrix A and an $n \times p$ B we can multiply them



$$c_{ij} = \boxed{\text{row } i}\ \boxed{\text{col } j} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} \cdot b_{nj}$$

Like page 173 question 3:

Let $C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 3 \end{bmatrix}$

$E = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$ $F = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Which of the following cannot be computed?

a. 2C - 3D

b. EF

c. EC

d. CE No

e. FE

f. 3 + C No

Example Are these equal?

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 11 & 4 & 13 \end{bmatrix}$$

Like page 173 question 19.
Write the system in the matrix form $Ax = b$.
Then find the solution in vector form as in equation (9).

$$
\begin{aligned}
x1 \qquad\qquad\quad + 3x4 - x5 &= 2\\
x2 \qquad - 2x4 + 6x5 &= -1\\
x3 + x4 - 8x5 &= 5
\end{aligned}
$$

Solution:

$$
\underbrace{\begin{bmatrix} 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & -2 & 6 \\ 0 & 0 & 1 & 1 & -8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}}_{b}
$$

Here x_4, x_5 are free variables

$$x_3 = 5 - x_4 + 8x_5$$

$$x_2 = -1 + 2x_4 - 6x_5$$

$$x_1 = 2 - 3x_4 + x_5$$

Solution:

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 3x_4 + x_5 \\ -1 + 2x_4 - 6x_5 \\ 5 - x_4 + 8x5 \\ x4 \\ x5 \end{bmatrix}
$$

$$
= \begin{bmatrix} 2 \\ -1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -6 \\ 8 \\ 0 \\ 1 \end{bmatrix}
$$

This is the desired form.

Page 174 question 29

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A^2 = (a+d)A - (ad-bc)I

Thus A^2 - trace(A)A + (det A) I = 0
(a special case of the Cayley-Hamilton theorem).

Solution.

The left hand side is $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} a^2+bc & ab+ad \\ ac+cd & bc+d^2 \end{bmatrix}$$

The right side is $(a+d)A - (ad-bc)I$

$$= \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} - \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

These are equal.

$$\text{trace}(A) = a+d$$

$$\det(A) = ad - bc$$

Page 174 question 35.
Use the formula of Problem 29
(A^2 - trace(A)A + (det A) I = 0)
to find a 2 x 2 matrix A such that $A \neq 0$
and $A \neq I$ but such that A^2 = A.

$A^2 = (a+d)A - (ad-bc)I$

We want $A^2 = A$

Take $a+d = 1$, $ad - bc = 0$

e.g. $a = 1$ $d = 0$ $c = 0$ $b = 2$

$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ or many other possibilities